United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Cayley paper

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. The numbers 62, 63, 64, 65, 66, 67, 68, 69 and 70 are divided by, in some order, the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9, resulting in nine integers. The sum of these nine integers is S. What are the possible values of S?

Solution

The only multiple of nine in the first list is 63 and so this must be divided by 9, yielding a quotient of 7.

The only multiple of eight in the first list is 64 and so this must be divided by 8, yielding a quotient of 8.

The only remaining multiple of seven is 70 and so this must be divided by 7, yielding a quotient of 10.

Similarly we must have 66 divided by 6, 65 divided by 5 and 68 divided by 4, yielding quotients of 11, 13, and 17 respectively.

The only remaining multiple of three is 69 and so this must be divided by 3, the only remaining multiple of two is 62 and so this must be divided by 2, yielding quotients of 23 and 31 respectively.

Finally we must have 67 divided by 1, yielding a quotient of 67.

This means that the value of S is unique and $S = 7 + 8 + 10 + 11 + 13 + 17 + 23 + 31 + 67 = 187$.

2. A palindromic number is a positive integer which reads the same when its digits are reversed, for example 269 962. Find all six-digit palindromic numbers that are divisible by 45.

SOLUTION

Let P represent such a six-digit number.

As P is palindromic it must have the form '$abccba$', where a, b and c are single-digits and a is non-zero (as P must be a six-digit number).

As P is divisible by 45, it must also be divisible by 5 which means that $a = 5$ and so P can be written as '$5bccb5$'.

P must be divisible by 9 and so the sum of its digits must also be divisible by 9. This means that $10 + 2b + 2c$ must be a multiple of 9. Dividing the previous expression by 2 must still yield a multiple of 9 (as 2 and 9 are co-prime), and so $5 + b + c$ must be a multiple of nine.

We next focus on the possible values of b.

If $b = 0$ then $5 + 0 + c$ must be a multiple of 9 and so c must be 4.

Similarly, we have:

If $b = 1$ then c must be 3.

If $b = 2$ then c must be 2.

If $b = 3$ then c must be 1.

If $b = 4$ then c must be 0 or 9.

If $b = 5$ then c must be 8.

If $b = 6$ then c must be 7.

If $b = 7$ then c must be 6.

If $b = 8$ then c must be 5.

If $b = 9$ then c must be 4.

The possible values of P are 504405, 513315, 522225, 531135, 540045, 549945, 558855, 567765, 576675, 585585 and 594495.

3. Consider the equation $0.abcd + 0.efgh = 1$, where each letter stands for a digit from 1 to 8 inclusive.

(a) Suppose each letter stands for a different digit. Prove that there are no solutions.

(b) Suppose instead that digits may be repeated. How many solutions are there? (You may give your final answer as a product of prime numbers if you wish.)

Note that $(0.abcd, 0.efgh)$ *and* $(0.efgh, 0.abcd)$ *are considered to be the same solution.*

SOLUTION

(a) By writing 1 as 1.0000 we can see that $d + h$ must end in a zero, and as both d and h are single digits less than or equal to 8 this means that $d + h = 10$.

We must also have $c + g + 1 = 10$, where the 1 is due to the carry from $d + h$.

Similarly, we must have $b + f + 1 = 10$ and $a + e + 1 = 10$.

To summarise, after simplification we have that $d + h = 10$, $c + g = 9$, $b + f = 9$ and $a + e = 9$.

The possibilities for pairs of digits which sum to 9 are $(1, 8)$, $(2, 7)$, $(3, 6)$ and $(4, 5)$.

No matter which three of these pairs we choose the two remaining integers (from the set 1 to 8) will also sum to 9 and so we cannot satisfy the equation $d + h = 10$. This means that there are no solutions to the given equation, if the digits are to be different.

(b) The equations derived in part (a) still hold but we may now repeat a digit, which we know is necessary for finding solutions to the given equation.

The pair of digits (d, h) must sum to 10 and so the possibilities are:

$(2, 8)$, $(3, 7)$, $(4, 6)$, $(5, 5)$, $(6, 4)$, $(7, 3)$ and $(8, 2)$.

Each of the pairs of digits (c, g), (b, f) and (a, e) must sum to 9 and so the possibilities are:

$(1, 8)$, $(2, 7)$, $(3, 6)$, $(4, 5)$, $(5, 4)$, $(6, 3)$, $(7, 2)$ and $(8, 1)$.

As listed above, for the pair (d, h) there are 7 choices and for each of the pairs (c, g), (b, f) and (a, e) there are 8 choices. By the product rule for counting, we can choose four relevant pairs in $7 \times 8 \times 8 \times 8 = 7 \times 2^9 = 3584$ ways.

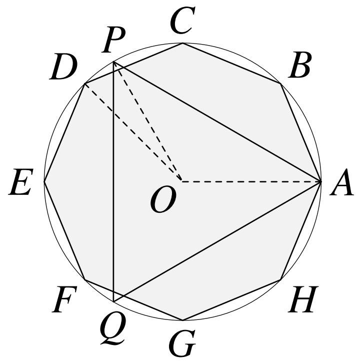
This is not the number of solutions to the given equation because we have overcounted by a factor of two. This is because in making a choice for each of the pairs listed above we have counted, for example, 0.1234+0.8766 and 0.8766+0.1234 as distinct solutions. The question tells us that these are not distinct solutions.

This means that the number of solutions to the given equation is $7 \times 2^8 = 1792$.

4. The regular octagon $ABCDEFGH$ is inscribed in a circle. Points P and Q are on the circle, with P between C and D, such that APQ is an equilateral triangle. It is possible to inscribe a regular n-sided polygon, one of whose sides is PD, in the circle. What is the value of n?

SOLUTION

Given that $ABCDEFGH$ is a regular octagon, and letting its centre be O, we find that $\angle AOD = 3 \times 360° \div 8 = 135°$.



Similarly, given that APQ is an equilateral triangle, we also have $\angle AOP = 360° \div 3 = 120°$.

Therefore the angle $POD = 135° - 120° = 15°$. Now, since 15 is factor of 360, it is true that PD is a side of a regular polygon inscribed in the circle. The number of sides, n, of this polygon is $360 \div 15 = 24$.

5. Consider equations of the form $ax + b = c$, where a, b and c are integers such that one is the sum of the other two and a is non-zero. What are the possible integer values of x?

SOLUTION

We have three cases to consider: $c = b + a$, $b = c + a$ and $a = b + c$.

For $c = b + a$:

In this case the given equation can be written as $ax + b = b + a$, which can be simplified to give $ax = a$. As we were told in the question that a is non-zero we may divide the previous equation by a to give $x = 1$.

For $b = c + a$:

In this case the given equation can be written as $ax + c + a = c$, which can be rearranged and simplified to give $ax = -a$. Again, as we were told in the question that a is non-zero we may divide the previous equation by a to give $x = -1$.

For $a = b + c$:

In this case the given equation can be written as $(b + c)x + b = c$, which can be rearranged to give $x = \frac{c-b}{b+c}$.

This expression for x is more complicated than the previous two cases and it is not clear what integer values it can take.

To simplify things we will let $a = 2$. This means that $b + c = 2$, which can be rearranged to give $c = 2 - b$.

We can now rewrite the expression for x in terms of b only, giving $x = \frac{2-b-b}{2}$ which after simplifying leads to $x = 1 - b$.

We may set b to be any integer we wish, which means that $x = 1 - b$ can also equal any integer we wish.

If we want $x = n$, then we set $a = 2$, $b = 1 - n$ and $c = 1 + n$. Each of these three expressions will give integers if n is an integer and they satisfy $a = b + c$.

The result of the final case means we know that under the conditions given in the question, any integer is a possible value for x.

6. Seth has nine stones: three painted blue, three painted red and three painted yellow. The blue stones are labelled 1, 2 and 3, as are the red stones and the yellow stones. He builds a vertical tower with three stones, putting one on top of another.

Three stones form a *set* if any of the following hold:

(i) They all have the same colour;

(ii) They are all labelled with the same number;

(iii) They all have different colours;

(iv) They are all labelled with different numbers.

In how many ways can he build a tower that **avoids** creating a set?

SOLUTION

We can describe a particular tower by inserting B (Bottom), M (Middle) and T (Top) into three of the nine empty cells in the table on the right:

	Blue	Red	Yellow
1			
2			
3			

For example, the tower made up of all blue stones, with the number 1 at the bottom, 2 in the middle and 3 at the top would look like:

	Blue	Red	Yellow
1	B		
2	M		
3	T		

In order to avoid a set as described in the question, when three empty cells are chosen we must omit exactly one row and exactly one column. Otherwise there would be one of each colour or one of each number in the tower and a set would be formed.

There are 3 choices for the omitted row and 3 choices for the omitted column. Once a row and a column have been omitted there remain four empty cells, of which we must choose three.

For example, if we omit the second row and second column, we must choose three of the four unshaded cells in the table on the right:

	Blue	Red	Yellow
1		(shaded)	
2	(shaded)	(shaded)	(shaded)
3		(shaded)	

No matter which row and column are omitted, we can choose three of the four remaining cells by instead choosing one of the cells to omit. There are 4 choices for which cell to omit.

We then have to decide how to arrange B, M and T into these three cells. There are 3 choices for where to place B, then 2 choices for where to place M and finally a single choice for where to place T.

By the product rule for counting, we have that the number of towers which avoid a set is $3 \times 3 \times 4 \times 3 \times 2 \times 1 = 216$.